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Characteristic polynomials of some graph coverings

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Abstract

We give a formula for the characteristic polynomial of the derived graph covering of a graph with voltages in any finite group.

Graphs treated here are finite simple graphs. Let G be a graph and $A(G)$ its adjacency matrix. Then the *characteristic polynomial* $\Phi(G;\lambda)$ of G is defined by $\Phi(G;\lambda)=\det(\lambda I - A(G))$. The eigenvalues of $A(G)$ are called the *eigenvalues* of G. Schwenk [6] studied relations between the characteristic polynomials of some related graphs. Kitamura and Nihei [4] discussed the structure of regular double coverings of graphs by using their eigenvalues. Chae *et al.* [2] gave the complete computations of the characteristic polynomials of K_2 (or $\bar{K}_2$)-bundles over graphs. Kwak and Lee [5] computed the characteristic polynomial of a graph bundle when its voltage assignment takes in an abelian group. Sohn and Lee [7] introduced weighted graph bundles and showed that the characteristic polynomial of a weighted $K_2(\bar{K}_2)$-bundles over a weighted graph G_w can be expressed as a product of characteristic polynomials of two weighted graphs whose underlying graphs are G. Furthermore, they gave the signature of a link whose corresponding weighted graph is a double covering of that of a given link. In this paper, we establish an explicit decomposition formula for the characteristic polynomial of the derived graph covering of a graph with voltages in any finite group.

Let $D(G)$ be the arc set of the symmetric digraph corresponding to G and Γ a finite group. Then a mapping $\alpha: D(G)\to\Gamma$ is called an *ordinary voltage assignment* if $\alpha(v,u)=\alpha(u,v)^{-1}$ for each $(u,v)\in D(G)$. The pair (G,α) is called an *ordinary voltage*

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graph. The *derived graph* G^α of the ordinary voltage graph (G,α) is defined as follows:

$$V(G^\alpha)=V(G)\times\Gamma,\quad E(G^\alpha)\subseteq E(G)\times\Gamma,$$

and $(u,g),(v,h)$ are adjacent in G^α if and only if $uv\in E(G)$ and $h=g\alpha(u,v)$. The graph G^α is called a *derived graph covering of* G *with voltages* in Γ.

For propositions concerning the representation of groups the reader is referred to [1]. For a square matrix B, we define $\Phi(B;\lambda):=\det(\lambda I-B)$.

The block diagonal sum $M_1\dotplus\cdots\dotplus M_s$ of square matrices $M_1,\ldots,M_s$ is defined as the square matrix

$$\begin{pmatrix} M_1 & & 0\\ & \ddots & \\ 0 & & M_s\end{pmatrix}.$$

If $M_1=\cdots=M_{a_1}=N_1$, $M_{a+1}=\cdots=M_{a_1+a_2}=N_2,\ldots,M_{s-a_t+1}=\cdots=M_s=N_t$, we write $M_1\dotplus M_2\dotplus\cdots\dotplus M_s=a_1\circ N_1\dotplus a_2\circ N_2\dotplus\cdots\dotplus a_t\circ N_t$.

Theorem 1. *Let G be a graph, Γ a finite group and $\alpha:D(G)\to\Gamma$ an ordinary voltage assignment. Furthermore, let $\rho_1=1,\rho_2,\ldots,\rho_h$ be the irreducible representations of Γ, and f_i the degree of ρ_i for each i, where $f_1=1$. For $g\in\Gamma$, the matrix A_g is defined as follows:*

$$A_g=(a_{uv}^{(g)}),\qquad a_{uv}^{(g)}=\begin{cases}1 & \text{if } \alpha(u,v)=g \text{ and } (u,v)\in D(G),\\ 0 & \text{otherwise.}\end{cases}$$

Then one has

$$\Phi(G^\alpha;\lambda)=\Phi(G;\lambda)\prod_{j=2}^{h}\left\{\Phi\left(\sum_{g\in\Gamma}\rho_j(g)\otimes A_g;\lambda\right)\right\}^{f_j},$$

where $\otimes$ is the Kronecker product of matrices.

Proof. Set $V(G)=\{v_1,\ldots,v_n\}$ and $\Gamma=\{g_1=1,g_2,\ldots,g_m\}$. Arrange the vertices of G^α in m blocks;

$$(v_1,1),\ldots,(v_n,1);(v_1,g_2),\ldots,(v_n,g_2);\ldots;(v_1,g_m),\ldots,(v_n,g_m).$$

For $g\in\Gamma$, the matrix $P_g=(p_{ij})$ is defined as follows:

$$p_{ij}=\begin{cases}1 & \text{if } g_ig=g_j,\\ 0 & \text{otherwise.}\end{cases}$$

Then we have

$$A(G^\alpha)=\sum_{g\in\Gamma}P_g\otimes A_g.$$

Let ρ be the right regular representation of Γ. Then we have $\rho(g) = P_g$ for $g \in \Gamma$. Furthermore there exists a regular matrix P such that

$$P^{-1}\rho(g)P = (1) \dot{+} f_2 \circ \rho_2(g) \dot{+} \cdots \dot{+} f_h \circ \rho_h(g) \quad \text{for each } g \in \Gamma.$$

Putting

$$B = (P^{-1} \otimes I_n) A(G^\alpha)(P \otimes I_n)$$

we have

$$B = \sum_{g \in \Gamma} \{(1) \dot{+} f_2 \circ \rho_2(g) \dot{+} \cdots \dot{+} f_h \circ \rho_h(g)\} \otimes A_g.$$

Therefore it follows that

$$\Phi(G^\alpha;\lambda) = \Phi(B;\lambda) = \Phi(G;\lambda) \prod_{j=2}^{h} \left\{ \Phi\left(\sum_{g \in \Gamma} \rho_j(g) \otimes A_g; \lambda \right) \right\}^{f_j}. \qquad \square$$

Corollary 2. $\Phi(G;\lambda) \mid \Phi(G^\alpha;\lambda)$.

Let G be a graph, Γ a finite abelian group and Γ^* the character group of Γ. For the mapping $f: D(G) \to \Gamma^*$, a pair $G_f = (G,f)$ is called a *weighted graph*. Given any weighted graph G_f, the adjacency matrix $A(G_f) = (a_{f,uv})$ of G_f is the square matrix of order $|V(G)|$ defined by

$$a_{f,uv} = a_{uv} \cdot f(u,v).$$

The characteristic polynomial of G_f is that of its adjacency matrix, and is denoted $\Phi(G_f;\lambda)$ [7].

Corollary 3. *Let α be an ordinary voltage assignment on a graph G in a finite abelian group Γ. Then*

$$\Phi(G^\alpha;\lambda) = \prod_{\chi \in \Gamma^*} \Phi(G_{\chi \circ \alpha};\lambda).$$

Proof. Each irreducible representation of Γ is a linear representation, and these constitute the character group Γ^*. By Theorem 1, we have

$$\Phi(G^\alpha;\lambda) = \Phi(G;\lambda) \prod_{\chi \in \Gamma^* \setminus \{1\}} \Phi\left(\sum_{g \in \Gamma} \chi(g) A_g; \lambda \right).$$

Since $\sum_g \chi(g) A_g = A(G_{\chi \circ \alpha})$, it follows that

$$\Phi(G^\alpha;\lambda) = \prod_{\chi \in \Gamma^*} \Phi(G_{\chi \circ \alpha};\lambda). \qquad \square$$

Corollary 4 (Chae et al. [2, Theorem 4]; Kitamura and Nihei [4, Theorem 1]). *Let α be an ordinary voltage assignment on a graph G in the group $Z_2 = \{1, -1\}$. Then*

$$\Phi(G^\alpha;\lambda) = \Phi(G;\lambda)\Phi(G_\alpha;\lambda).$$

Proof. By Corollary 3, we have

$$\Phi(G^{\alpha};\lambda) = \Phi(G;\lambda)\,\Phi(G_{\chi\circ\alpha};\lambda),$$

where $\chi(1) = 1$ and $\chi(-1) = -1$. Since $(\chi\circ\alpha)(u,v) = \alpha(u,v)$ for each $(u,v) \in D(G)$, it follows that $\chi\circ\alpha = \alpha$. $\square$

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